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On the uniqueness of ϕ^4 interactions in two- and three-component spin systems

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Abstract. We examine the constraints imposed on the ϕ^4 interactions of a two- or three-component spin system by the demand that the spin–spin correlation function is isotropic. The phase transition behaviour of such systems is considered and proves to be essentially unique. The two-component model provides a very clean example of a redundant operator in the renormalization group formalism.

1. Introduction

Considerable attention has already been given to the conditions imposed upon the interactions of a field theory by the constraint that they must be consistent with a quadratic term (in the Hamiltonian) which is highly symmetric. The main application of this idea has been made in relativistic quantum field theory, with attempts to generate SU(n) symmetric Yukawa couplings between fermions and bosons from the demand that the renormalization effects of the Yukawa couplings be consistent with having the same mass for all the bosons, and another for all the fermions (see Sudarshan *et al* 1964, Fleischman *et al* 1967 and references therein).

The principal aim of this paper is to extend this work to the self-interactions of a set of boson fields. Our main interest in this problem arises not in relativistic quantum field theory but in the statistical mechanics of an interacting n-component Bose field $\phi_i(x)$ ($i = 1, \ldots, n$). The relevance of such models for many thermodynamic problems is well known; here we shall work always in terms of a magnetic system in which $\phi_i(x)$ is interpreted as a spin or magnetization density. The statement of equal boson masses translates in this language into an isotropic (zero field) susceptibility tensor $\chi_{ij} = \chi\delta_{ij}$.

This property is particularly important in the phase transition region of the spin system because it implies that the symmetry of the system ensures that all modes ϕ_i go critical (have infinite susceptibility) at the same temperature. Without this condition, in general only some of the eigenvalues of the susceptibility tensor will become infinite simultaneously and the behaviour of the system close to the critical temperature will effectively be that of a system with fewer than n spin components. (A general review of the renormalization group approach to phase transitions is given by Wilson and Kogut (1974). Treatment of the case of anisotropic susceptibility is given in this approach by Fisher and Pfeuty (1972) and Wegner (1972b); a more recent review of this problem is contained in the paper by Pfeuty *et al* (1974).)

We shall start from a theory with arbitrary ϕ^4 interactions, $u_{ijkl}\phi_i\phi_j\phi_k\phi_l$, specified by coupling constants u_{ijkl}. An alternative specification is given in terms of an arbitrary (complete) set of tensors $N^{\mu}{}_{ijkl}$ by writing $u_{ijkl} = u_{\mu}N^{\mu}{}_{ijkl}$ (the summation convention on repeated indices should always be understood). We do not consider interactions of $O(\phi^6)$; these are expected to be irrelevant for the phase transition behaviour in the sense that they do not modify exponents or the principal scaling functions (although they can produce secondary effects in (for example) transverse susceptibility below the critical temperature through their symmetry properties).

Many aspects of the critical behaviour of such a system with arbitrary ϕ^4 interactions have been considered by Brézin *et al* (1974). In this paper we enumerate the allowed interactions $N^{\mu}{}_{ijkl}$ which are consistent with an isotropic susceptibility for a two- or three-component field. The former is trivial to do, the latter is not, but the result is the same in both cases; only two independent interactions are permitted, the $O(n)$ symmetric one $(\phi^2)^2$ and $\phi_1^4 + \ldots + \phi_n^4$ ($n = 2$ or 3). The latter has cubic point-group symmetry which is well known to have only one invariant two-tensor, which is symmetric under interchange of indices, namely δ_{ij}.

The outline of the paper is as follows. In § 2 we derive the above result and review briefly the implications for phase transition behaviour. Section 3 contains a discussion of redundant variables (Wegner 1974) which arise very naturally in this problem. The $n = 2$ case provides a particularly clean example which illustrates a possible pitfall when including redundant variables.

2. Isotropy constraint

The problem of finding ϕ^4 terms consistent with isotropic susceptibility is essentially a group-theoretic one. It may be posed as: enumerate *all* the invariant (totally symmetric) four-tensors of *all* groups which have a real irreducible representation of dimension n. The resolution of this problem is less facile than the posing of it. Instead we adopt the more pragmatic approach that the ϕ^4 interactions must be consistent with an isotropic two-spin correlation function in perturbation theory. Thus we demand that all graphs with two external legs are proportional to δ_{ij}, order by order in perturbation theory. This will eventually provide sufficient equations to determine the allowed types of $N^{\mu}{}_{ijkl}$. This approach is rather weak in practice however for large n (many spin components) because, roughly speaking, the number of ϕ^4 couplings increases as n^4 whereas the number of constraints from a given two-point graph rises only as n^2. In this section we show that this constraint is sufficiently strong that, for the two cases $n = 2, 3$, only two essentially unique ϕ^4 couplings are allowed. In § 3 we show that this uniqueness property does not hold for $n \geqslant 4$.

For $n = 2$ there are only five fourth-rank tensors N_{ijkl} which are totally symmetric in the indices. The isotropy condition at the lowest order (figure 1) is just a trace condition (Brézin *et al* 1974): $N_{iijk} \propto \delta_{jk}$. This imposition eliminates two tensors. The

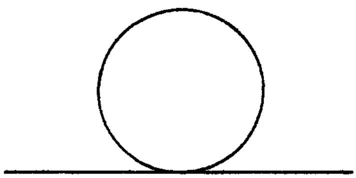

Figure 1.

three remaining candidates may be combined to form S, C and A, where each is defined by the interactions:

$$S_{ijkl}\phi_i\phi_j\phi_k\phi_l = \left(\sum_i \phi_i^2\right)^2 \qquad \text{`symmetric'} \tag{1a}$$

$$C_{ijkl}\phi_i\phi_j\phi_k\phi_l = \sum_i \phi_i^4 \qquad \text{`cubic'} \tag{1b}$$

$$A_{ijkl}\phi_i\phi_j\phi_k\phi_l = \phi_1^3\phi_2 - \phi_2^3\phi_1. \tag{1c}$$

A linear combination of these would form the most general interaction Hamiltonian consistent with lowest-order isotropy. A more convenient and equally general basis is to replace (1*b*) by a traceless counterpart:

$$T_{ijkl} \equiv C_{ijkl} - \tfrac{3}{4}S_{ijkl}. \tag{1d}$$

To investigate isotropy at higher order we consider the most general Hamiltonian, specified by (indices suppressed)

$$H_I(u_S, u_T, u_A; \phi) = (u_S S + u_T T + u_A A)(\phi^4/4!). \tag{2}$$

Without specifically looking at higher-order diagrams, let us redefine fields by a rotation

$$\tilde{\phi}_i = R_{ij}(\alpha)\phi_j$$

with

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{3}$$

Clearly the kinetic and mass terms remain form invariant, while the interaction term becomes

$$H_I(\tilde{u}_S, \tilde{u}_T, \tilde{u}_A; \tilde{\phi}) \tag{4}$$

with

$$\tilde{u}_S = u_S \tag{5a}$$

and

$$\begin{pmatrix} \tilde{u}_T \\ \tilde{u}_A \end{pmatrix} = R(4\alpha)\begin{pmatrix} u_T \\ u_A \end{pmatrix} \tag{5b}$$

so that, by an appropriate choice of α, $\tilde{u}_A$ can be set to zero. Such a system is well known to have isotropic susceptibility at all orders.

The analysis for $n = 3$ is only technically more complicated. We now begin with 15 tensors of which one is S and five are eliminated by the trace condition. The remaining nine can be considered to be traceless tensors. To go beyond this lowest order we employ the convenient basis of the nine fourth-order ordinary spherical harmonics Y_{4m}, $m = -4, \ldots, 4$ (see eg Wegner 1972b). Explicitly we define 'spherical coordinates' by

$$\phi_1 \equiv \Phi \sin\theta\cos\psi \qquad \phi_2 \equiv \Phi\sin\theta\sin\psi \qquad \phi_3 \equiv \Phi\cos\theta \tag{6}$$

so that any traceless interaction may be represented by

$$u_m \Phi^4 Y_{4m}(\theta, \psi). \tag{7}$$

For example, in this basis the traceless cubic interaction $u_T T_{ijkl}\phi_i\phi_j\phi_k\phi_l$ (for $n = 3$, $T = C - \frac{3}{5}S$) is represented by $u_0(\frac{5}{14})^{1/2} = u_4 = u_{-4} \propto u_T$ and all others zero. Due to hermiticity of H_I, $u_m^* = (-1)^m u_{-m}$. Next we apply the rotations; for $n = 3$ there are three of these (eg the Euler angles). Using two of the rotations, it is always possible to bring a general interaction to a form with u_1 real and u_0 zero. These rotations are unitary transformations on the nine-dimensional space $\{u_m\}$ so that we can choose our normalization by $u_1 = 1$ for example. Without loss of generality our interaction Hamiltonian depends on u_S and the six parameters u_2, u_3, u_4 (each of the last three being complex). Now we impose the isotropy constraint at second order (figure 2).

$$u_{ijkl}u_{jklm} \propto \delta_{im}. \tag{8}$$

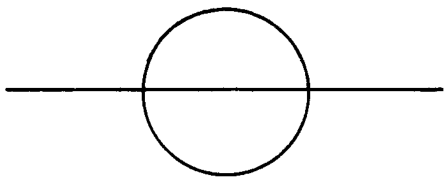

Figure 2.

Note first that if u were any linear combination of S and a traceless N which satisfied (8), then u also satisfies it. In effect we can neglect S completely and (8) represents five quadratic equations in the six unknowns (u_2, u_3, u_4). The equations turn out to be all independent and there is a unique one-parameter family of solutions which are related to each other by the third rotation. We thus arrive at the same conclusion as for the $n = 2$ case: any interaction satisfying the isotropy condition (at first and second order) can be brought into the standard symmetric-plus-cubic form by a redefinition of fields. Again, such a system is known to satisfy isotropy to all orders.

The significance of this 'uniqueness' property for critical phenomena may be seen best in the light of the multiplicative renormalization group formulation (Zinn-Justin 1975). A system of differential equations

$$\mathrm{d}u_\mu/\mathrm{d}\tau = \beta_\mu(u) \tag{9}$$

is set up in the space of couplings $\{u_\mu\}$. The fixed points of these equations (u^* with $\beta(u^*) = 0$) and their associated stability matrices $(\partial\beta_\mu|\partial u_\nu|_{u=u^*})$ govern the behaviour of the spin system undergoing phase transition. If such a system specified by a point in coupling space lies in the domain of stability of a certain fixed point, then its behaviour in the critical region (eg exponents) can be obtained through the stability matrix of that fixed point only. In particular, for a system with S and C interactions only, there are four fixed points (in the ϵ-expansion) and four sets of exponents associated with these (Wilson and Fisher 1972, Aharony 1973). *A priori*, investigations of systems with other types of interactions should lead to other fixed points and therefore, possibly, other exponents. However, for $n = 2$ or 3, since we can bring a system with arbitrary interactions into the $S + C$ form by a rotation of the fields, all other fixed points are equivalent to one of the above four and there are no new features in critical behaviour.

3. Discussion and conclusion

The results of § 2 are summarized by the statement that two- and three-component spin systems with isotropic susceptibility have essentially only the symmetric and cubic ϕ^4 interactions in equations (1*a*) and (1*b*); all other ϕ^4 interactions are equivalent to these two by an appropriate redefinition of fields such as in equation (3).

For systems with $n \geqslant 4$ spin components this result is no longer true. Two counter examples can be given:

(i) The ϕ^4 interactions of the generalized Potts model (Potts 1952) consist of the symmetric term (equation (1*a*)) and a term

$$\sum_{p=1}^{n+1} t_i^p t_j^p t_k^p t_l^p \phi_i \phi_j \phi_k \phi_l$$

where t^p is the set of $n+1$ vectors defining the $n+1$ vertices of the hypertetrahedron in n dimensions. This model is multiplicatively renormalizable (no new couplings being introduced by renormalization effects), is consistent with isotropic susceptibility and, for $n \geqslant 4$, is not equivalent to the symmetric/cubic system (Zia and Wallace 1975).

(ii) For $n = 4$ there is a third independent coupling $\lambda\phi_1\phi_2\phi_3\phi_4$ which, when combined with the symmetric and cubic couplings, produces a multiplicatively renormalizable system. In $\{u_S, u_C, \lambda\}$ space there is an interesting configuration of renormalization group fixed points. They lie on the three *planes* ($\lambda = 0, 2\lambda = \pm u_C$) which are mapped (1–1) into each other by discrete O(4) rotations of the fields. On the other hand no other region in the (3-d) space is related to these planes by rotation because, otherwise, there will be more fixed points. That any system specified in this full space is consistent with isotropic susceptibility can be easily checked. It is conjectured that systems not lying in the three planes will undergo a first-order phase transition, but further discussion seems inappropriate here.

In mathematical terms these examples show that algebraic varieties produced in the coupling constant space by the isotropic susceptibility constraint can become rather complicated for $n \geqslant 4$.

It is perhaps worth clarifying one point. For two- or three-component spin systems we have enumerated the only ϕ^4 couplings which are completely consistent with isotropic susceptibility (of course these theories do not have isotropic susceptibility in the region of spontaneous symmetry breaking, $T < T_c$). One may further ask if there are other ϕ^4 interactions which have a negligibly small effect on the leading singularities of the susceptibility tensor *in the critical region*. In the language of the renormalization group this is equivalent to enumerating the additional ϕ^4 interactions with respect to which the isotropic/cubic system is stable in the renormalization group sense. For $n \lesssim 3$ the most stable fixed point of the isotropic/cubic system is the isotropic Heisenberg one (Ketley and Wallace 1973, Nickel 1975). Now perturbations away from this isotropic fixed point are either fourth-order spherical harmonics or second-order spherical harmonics ($u_{ijkl}\phi^i\phi^j\phi^k\phi^l$ or $u_{ij}\phi^i\phi^j$ (ϕ^2) with $u_{ijkl} = 0, u_{ii} = 0$) or symmetric $(\phi^2)^2$. Each of these three types of perturbation is an eigenvector of the renormalization group with a unique eigenvalue for each type (Wegner 1972b, Wallace and Zia 1975). The isotropic fixed point is known to be stable for $n \lesssim 3$ to each of these perturbations. (See Brézin *et al* (1974) for the second-order spherical harmonic; stability properties have also been studied explicitly for some special cases by Nelson *et al* (1974).) Thus for $n \lesssim 3$ *any* ϕ^4 perturbations small enough to be in the domain of stability of the

Heisenberg fixed point will produce a susceptibility tensor whose leading singularities in the critical region are isotropic.

Finally let us return to the two-component system and consider the effect of including the physically redundant interaction (1*c*) as well as the symmetric and cubic terms (1*a*) and (1*d*). As was pointed out, the field transformation (3) induces a transformation in the space of interaction Hamiltonians $H_I(u_S, u_T, u_A, \phi)$ according to equation (5). This system provides a very simple example of the result that transformations induced in the space of interactions by field transformations leave invariant the renormalization group equations (compare Jona-Lasinio 1973, Wegner 1974), ie the β's in equations (9) are invariant under (5). (The equations (9) in common usage refer to the renormalized coupling constants rather than the bare ones which appear in the Hamiltonian itself; the transformations (5) are then the corresponding renormalized coupling constants. In the remainder of this article coupling constants should be understood as renormalized coupling constants.)

For example if one calculates the β functions up to one loop in perturbation theory using the multiplicative renormalization group (Zinn-Justin 1975), one finds

$$\dot{u}_S = -\epsilon u_S + (a/2)[\tfrac{10}{3}u_S^2 + \tfrac{3}{8}(u_T^2 + u_A^2)] \qquad (11a)$$

$$\dot{u}_T = -\epsilon u_T + 2au_S u_T \qquad (11b)$$

$$\dot{u}_A = -\epsilon u_A + 2au_S u_A \qquad (11c)$$

where a is a constant of order 1. These equations are invariant with respect to (5). The resulting fixed-point equations are degenerate and one obtains the circle of fixed points

$$u_S^* = \epsilon/2a, \qquad u_T^{*2} + u_A^{*2} = 4\epsilon^2/9a^2 \qquad (12)$$

as shown in figure 3. This is a one-dimensional manifold of equivalent fixed points. The trajectories defined by the solutions of equation (11) lie in planes containing the u_S axis. As discussed by Wilson and Fisher (1972), in the $u_A = 0$ plane there are two equivalent fixed points corresponding to two decoupled Ising systems; they are the points $u_T = \pm 2\epsilon/3a$ on the circle (12).

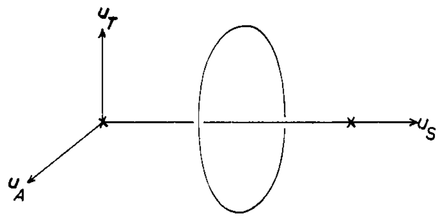

Figure 3.

This simple example illustrates clearly the appearance of manifolds of equivalent fixed points whenever physically redundant couplings are allowed in the Hamiltonian. This problem was first raised by Wilson (Wilson and Kogut 1974, appendix) and Hubbard and Schofield (1972) and discussed generally by Wegner (1974). The problem of redundant couplings is not a serious one in the multiplicative renormalization group; they are readily spotted and simply discarded. This is not the case in the 'functional formulation' of the renormalization group (Wilson and Kogut 1974, Wegner 1972a) where the Hamiltonian is allowed to be an arbitrary functional of the field ϕ. Such a system has a huge number of redundant couplings since it permits general redefinition of the field $\phi \to f(\phi)$.

We finish with a cautionary remark illustrated by the two-component model: if one allows redundant variables so that there is a manifold of equivalent fixed points, then it is certainly not clear *in general* how to avoid the situation in which the effective coupling constants end up in a limit cycle (or other recurrent behaviour) in the space of equivalent fixed points. Indeed it is easy to construct a renormalization group transformation in the two-component model which will achieve this, namely instead of relating the renormalized to the bare vertex functions by a wavefunction renormalization $Z^{1/2}(u)\,\delta_{ij}$ for each external line, allow a matrix wavefunction renormalization $Z^{1/2}(u)R_{ij}(\alpha(\ln\mu))$ where R_{ij} is a rotation as in equation (3) and μ is the renormalization point. (This is the analogue of the class $\psi_q\{S\}$ of renormalization group transformations considered by Wegner (1974); the renormalization group equations for matrix wavefunction renormalization are noted by Brézin *et al* (1974).) One then finds that equations (11) are replaced by

$$\dot{u}_S = -\epsilon u_S + (a/2)[\tfrac{10}{3}u_S^2 + \tfrac{3}{8}(u_T^2 + u_A^2)] \tag{13a}$$

$$\dot{u}_T = -\epsilon u_T + 2au_S u_T - 4\dot{\alpha} u_A \tag{13b}$$

$$\dot{u}_A = -\epsilon u_A + 2au_S u_A + 4\dot{\alpha} u_T. \tag{13c}$$

If $\dot{\alpha} \neq 0$ these equations have no fixed points with $u_S \neq 0$. In particular if $\dot{\alpha}$ = constant, what was previously a circle of equivalent fixed points (equation (12)) becomes a (unstable) limit cycle for the system (13).

Now it is clear in this example that the renormalization group transformation which produces this behaviour is a pretty stupid one to use, because for example the direction of the easy axes of the system will rotate as one changes the renormalization point. However it is certainly not as easy to see how to avoid wandering in the space of equivalent fixed-point Hamiltonians when the redundancy permits general field transformations of the form $\phi \to f(\phi)$.

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